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Approximate expression for the electric potential around a dust grain in isotropic collisionless plasma

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Introduction

Exact Solution

The nonlinear Poisson equation

$$\Delta\varphi = -4\pi e\,(n_\mathrm{i} - n_\mathrm{e})$$

with the charge density derived from the solution of the Vlasov equations

The expressions for $n_{\mathrm{i,e}}(r,\varphi)$ can be found in

T. Bystrenko and A. Zagorodny
Physics Letters A **299**, 383 (2002)

X.-Z. Tang and G. L. Delzanno
Phys. Plasmas **21**, 123708 (2014)

Model Potential

$$\beta = (a/\lambda_\mathrm{Di})(e\varphi_a/kT_\mathrm{e})(T_\mathrm{e}/T_\mathrm{i})$$

(nonlinearity parameter)

$$\varphi = \varphi_\mathrm{DH} + \varphi_\mathrm{as} \qquad \varphi_\mathrm{as} \sim (a/r)^{-2}$$

- $\beta \ll 1$ (linear regime)

$$\varphi_\mathrm{DH} \sim (a/r)\,e^{-r/\lambda_\mathrm{D}}$$

- $\beta \gg 1$ (nonlinear regime)

$$\varphi_\mathrm{DH} \sim (a/r)\,e^{-r/\lambda_s}$$

$$\lambda_s \gtrsim \lambda_\mathrm{D} \quad r \lesssim \lambda_\mathrm{D}$$

J. E. Daugherty et al.
J. Applied Physics **72**, 3934 (1992)

$$\Delta\varphi = -4\pi e\left[n_{\rm i}(r,\varphi) - n_{\rm e}(r,\varphi)\right]$$

If the following notations are used

$$x = r/a, \quad \tau_{\rm e} = T_{\rm e}/T_{\rm i}, \quad \varphi_{\rm e} = -e\,\varphi/kT_{\rm e}, \quad \varphi_{\rm i} = \varphi_{\rm e}\,\tau_{\rm e},$$

the ion number density can be written as $n_{\rm i} = n_1 + n_2$ with

$$n_1/n_0 = \sqrt{\varphi_{\rm i}/\pi} + (1/2)\,\mathrm{erfcx}\left(\sqrt{\varphi_{\rm i}}\right)$$

$$n_2/n_0 = \sqrt{f_{\rm i}/\pi} + (1/2)\,\mathrm{erfcx}(\sqrt{f_{\rm i}/g})\sqrt{g}, \quad \text{for} \quad f_{\rm i} \geq 0$$

$$n_2/n_0 = (1/2)\,\exp\left(\varphi_{\rm i}\right)\sqrt{g}, \quad \text{for} \quad f_{\rm i} < 0$$

and the electron number density is given by

$$n_{\rm e}/n_0 = (1/2)\,\exp(-\varphi_{\rm e})\Big[1 + \mathrm{erf}\left(\sqrt{f_{\rm e}}\right) + \\ + \exp\left(f_{\rm e}/gx^2\right)\mathrm{erfc}\left(\sqrt{f_{\rm e}/g}\right)\sqrt{g}\Big].$$

Here $g = 1 - x^{-2}$, $f_{\rm i} = \varphi_{\rm i} - \alpha_{\rm e}\tau_{\rm e}x^{-2}$, $f_{\rm e} = \alpha_{\rm e} - \varphi_{\rm e}$ and $\alpha_{\rm e} = e\varphi_a/kT_{\rm e}$

If the following notations are used

$$x = r/a, \quad y = r - a, \quad \varphi_\mathrm{e} = -e\,\varphi/kT_\mathrm{e}, \quad \alpha_\mathrm{e} = \varphi_\mathrm{e}(a), \quad \tau_\mathrm{e} = T_\mathrm{e}/T_\mathrm{i},$$

the model potential can be written as

$$\varphi_\mathrm{e}(r) = \frac{\hat{\alpha}_\mathrm{e}}{x}\, e^{-k_s\, y} + \varphi_\mathrm{as}(r), \quad \hat{\alpha}_\mathrm{e} = \alpha_\mathrm{e} - \varphi_\mathrm{as}(a),$$

where, $k_s = \lambda_s^{-1}$ is the inverse effective screening length and

$$\varphi_\mathrm{as}(r) = \omega_0\,(ak_0)^2\,\mathcal{F}(rk_0), \quad k_0 = \lambda_\mathrm{D}^{-1}, \quad \omega_0 = (1 + 2\alpha_\mathrm{e}\tau_\mathrm{e})\,/4(1+\tau_\mathrm{e})$$

Here $\mathcal{F}(\xi) = [e^{-\xi}\mathrm{Ei}(\xi) - e^{\xi}\mathrm{Ei}(\xi)]/(2\xi), \quad \mathcal{F}(\xi) \approx \xi^{-2} \;\; \text{for} \;\; \xi \gg 1$

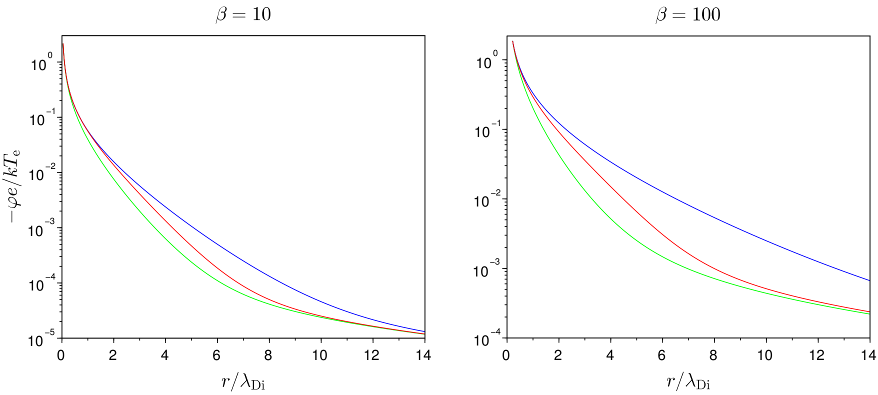
- $\lambda_s/\lambda_\mathrm{Di} = 1 + 0.2\sqrt{\beta}, \quad \beta \lesssim 10$ (B. Klumov, 2006)
- $\lambda_s/\lambda_\mathrm{Di} = 1 + 0.013\beta + 0.105\sqrt{\beta}, \quad \beta \lesssim 100$ (S. Khrapak, 2009)

Overview of the existing models
Electric potential distribution

- Exact solution (red)
- Solution of the linearized Poisson equation (green)
- Model potential with an effective screening length (blue)

$\beta = 10 \quad T_e/T_i = 100 \quad a/\lambda_{Di} \approx 0.05$

$\beta = 100 \quad T_e/T_i = 250 \quad a/\lambda_{Di} \approx 0.2$



$\lambda_s/\lambda_{Di} = 1 + 0.2\sqrt{\beta}$

$\lambda_s/\lambda_{Di} = 1 + 0.013\beta + 0.105\sqrt{\beta}$

If the following notations are used

$$x = r/a, \quad y = r - a, \quad \varphi_{\mathrm{e}} = -e\,\varphi/kT_{\mathrm{e}}, \quad \alpha_{\mathrm{e}} = \varphi_{\mathrm{e}}(a),$$

the model potential is given by

$$\varphi_{\mathrm{e}}(r) = \frac{\hat{\alpha}_{\mathrm{e}}}{x}\, e^{-k(y)\,y} + \varphi_{\mathrm{as}}(r), \quad \hat{\alpha}_{\mathrm{e}} = \alpha_{\mathrm{e}} - \varphi_{\mathrm{as}}(a),$$

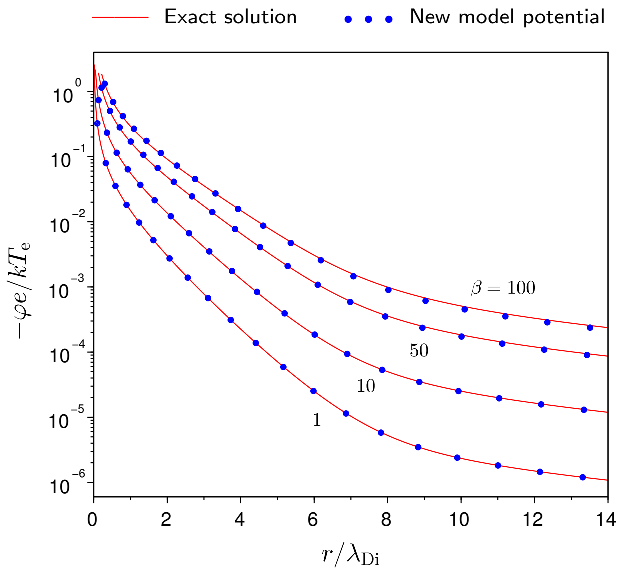
where

$$k(y) = k_* + (k_0 - k_*) f(c_* y), \quad f(\xi) = \ln(\cosh(\xi))/\xi, \quad k_0 = \lambda_{\mathrm{D}}^{-1}$$

$$k_*/k_0 = (1 + 0.65\sqrt{\beta}\,)^{-1/2}$$

$$c_*/k_0 = 0.58\,(1 + 1.2\sqrt{\beta}\,)^{-1/2}$$

Note that $f(\xi) \to 0, \;\; \xi \to 0$ and $f(\xi) \to 1, \;\; \xi \to \infty$



Exact solution
New model potential
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$-\varphi e/kT_{\mathrm{e}}$
$\beta = 100$
50
10
1
0
2
4
6
8
10
12
14
r/λ_{Di}

- Step I

The exact solution is rather well approximated by the potential

$$\varphi = \varphi_* + \varphi_{\mathrm{as}}$$

where φ_* is given by the solution of the Poisson equation with the following normalized charge density

$$\rho(\varphi_{\mathrm{i}}) = 2\sqrt{\varphi_{\mathrm{i}}/\pi} + \mathrm{erfcx}(\sqrt{\varphi_{\mathrm{i}}}) - \exp(-\varphi_{\mathrm{i}}/\tau_{\mathrm{e}})$$

Here $\varphi_{\mathrm{i}} = -e\varphi/kT_{\mathrm{i}}$ and $\tau_{\mathrm{e}} = T_{\mathrm{e}}/T_{\mathrm{i}}$. The function ρ is obtained from the analysis of the original expression near the maximum of $w = \rho\, r^2$.

- Step II

The right hand side of the Poisson equation can be written in DH form

$$-4\pi e n_0 \rho = k_\rho^2\, \varphi_{\mathrm{i}}, \quad k_\rho(\varphi_{\mathrm{i}}) = \lambda_{\mathrm{Di}}^{-1} \sqrt{\rho(\varphi_{\mathrm{i}})/\varphi_{\mathrm{i}}}$$

The inverse *effective screening length* is rather well approximated by

$$k_\rho \approx k_* + (k_0 - k_*) \tanh(c_* y), \quad y = r - a$$

- Step III

If the potential φ_* is written in the following DH form

$$\varphi_* \sim e^{-s(y)}/x, \quad x = r/a, \quad y = r - a,$$

we obtain the residual

$$\delta(\varphi_*) = \Delta\varphi_* - k_\rho^2\,\varphi_*, \quad \Delta\varphi_* = \varphi_*\Big[\,(\partial s/\partial y)^2 - \partial^2 s/\partial y^2\,\Big]$$

Using the condition $\delta(\varphi_*) = 0$ and neglecting the term $\partial^2 s/\partial y^2$ we obtain

$$\partial s/\partial y = k_\rho$$

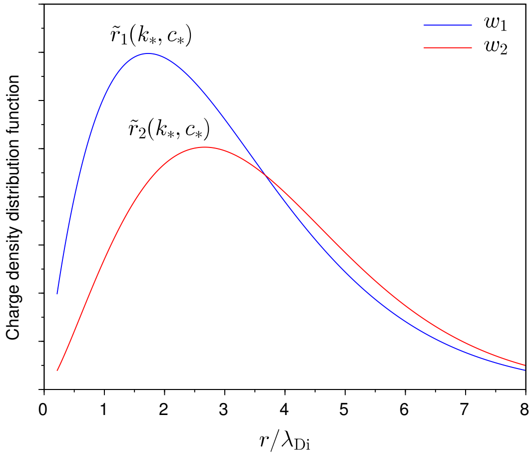
If we assume that $k_\rho = k_* + (k_0 - k_*)\tanh(c_* y)$ then we obtain

$$s(y) = y\Big[k_* + (k_0 - k_*)\,f(c_* y)\Big], \quad f(\xi) = \ln(\cosh(\xi))/\xi$$

$$\partial^2 s/\partial y^2 = c_*(k_0 - k_*)\Big[1 - \tanh(c_* y)^2\Big]$$

Note that $\partial^2 s/\partial y^2$ rapidly decreases with the distance

Parameters of the model



$$\delta(\varphi_*) = \Delta\varphi_* - k_\rho^2\,\varphi_*$$

Charge density distribution functions

$$w_1 = \Delta\varphi_*\,r^2$$

$$w_2 = k_\rho^2\,\varphi_*\,r^2$$

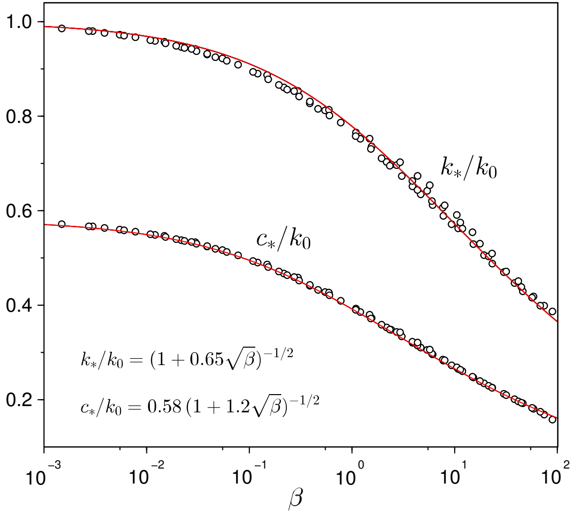
Parameter k_*:

$$\tilde{r}_1(k_*, c_*) = \tilde{r}_2(k_*, c_*)$$

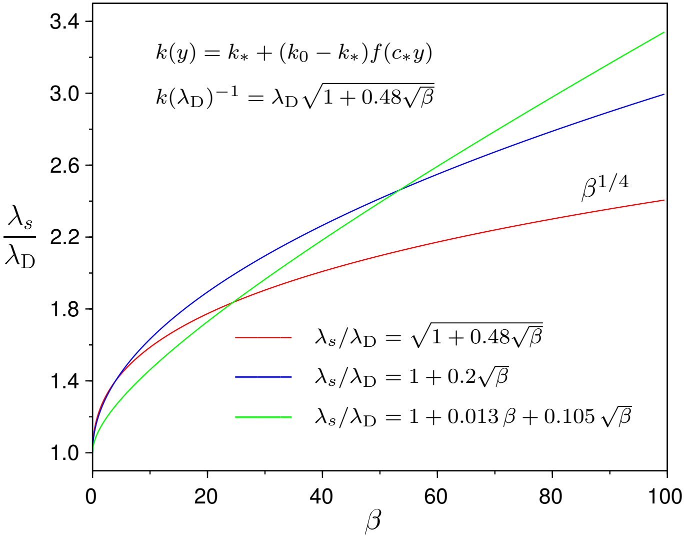
Parameter c_*:

$$\int_a^\infty (w_1 - w_2)\,dr = 0$$

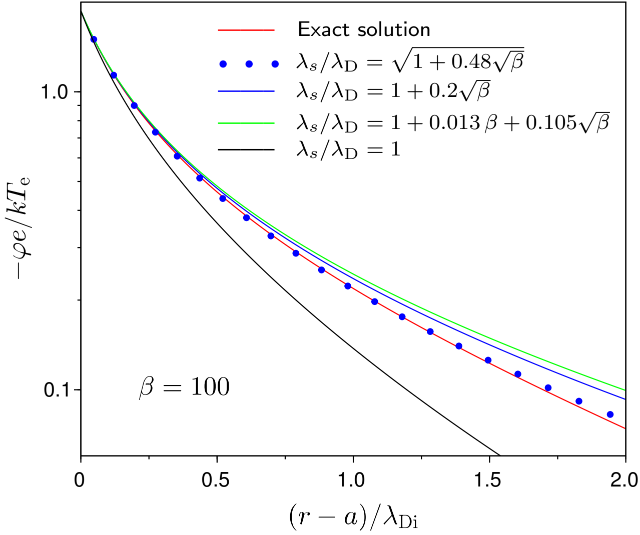
Parameters of the model
Results



Effective screening length



Effective screening length
Electric potential distribution

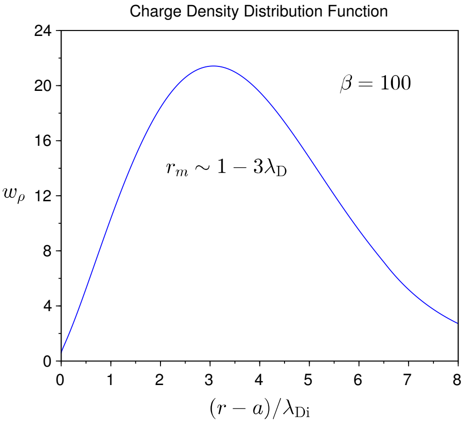


Conclusions

- A new approximate expression for the potential distribution around an absorbing dust grain in isotropic collisionless plasma is proposed
- The proposed model potential is shown to be in excellent agreement with the exact solution both in linear and non-linear regimes of screening (up to $\beta \sim 100$)
- The conditions of applicability of the model are estimated to be $a \lesssim 0.2\lambda_{\rm Di}$ and $T_{\rm e}/T_{\rm i} \gtrsim 10$

Appendix A

$$w_\rho = (\rho / e n_0)(r k_0)^2$$



$$x = r/a, \quad \tau_\mathrm{e} = T_\mathrm{e}/T_\mathrm{i},$$

$$\varphi_\mathrm{e} = -e\,\varphi/kT_\mathrm{e}, \quad \varphi_\mathrm{i} = \tau_\mathrm{e}\,\varphi_\mathrm{e},$$

$$\alpha_\mathrm{e} = e\varphi_a/kT_\mathrm{e}$$

$$x \gg 1, \quad g = 1 - x^{-2} \approx 1$$

$$\varphi_\mathrm{i} - \alpha_\mathrm{e}\tau_\mathrm{e}\,x^{-2} \approx \varphi_\mathrm{i}$$

$$n_\mathrm{i}/n_0 \approx 2\sqrt{\varphi_\mathrm{i}/\pi} + \mathrm{erfcx}(\sqrt{\varphi_\mathrm{i}})$$

$$n_\mathrm{e}/n_0 \approx \exp(-\varphi_\mathrm{e})$$

V. N. Tsytovich et al.
IEEE Trans. Plasma Sci. (2004)

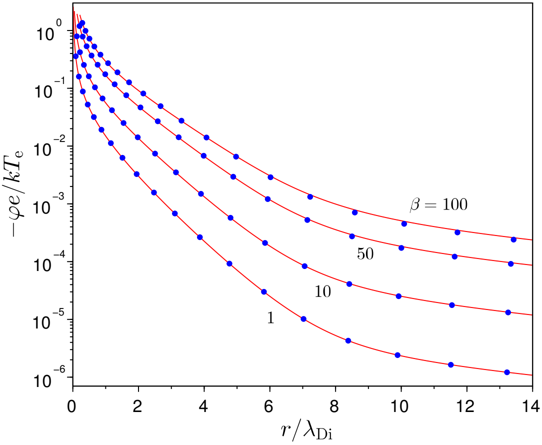
Appendix B

— Exact solution

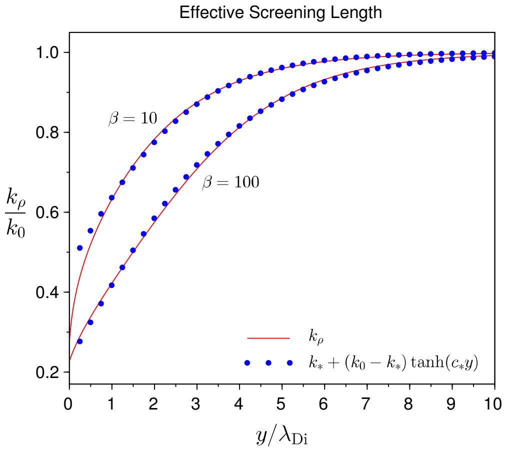
• • • $\varphi = \varphi^* + \varphi_{\rm as}$, where φ^*

is the solution of the Poisson equation with the charge density $-en_0\rho$

$\rho = 2\sqrt{\varphi_{\rm i}/\pi} + {\rm erfcx}(\sqrt{\varphi_{\rm i}}) - \exp(-\varphi_{\rm i}/\tau_{\rm e})$ $\qquad \varphi_{\rm i} = -e\,\varphi/kT_{\rm i} \quad \tau_{\rm e} = T_{\rm i}/T_{\rm e}$



$$\rho = 2\sqrt{\varphi_\mathrm{i}/\pi} + \mathrm{erfcx}(\sqrt{\varphi_\mathrm{i}}) - \exp(-\varphi_\mathrm{i}/\tau_\mathrm{e})$$



$$\varphi_\mathrm{i} = -e\,\varphi/kT_\mathrm{i} \qquad \tau_\mathrm{e} = T_\mathrm{i}/T_\mathrm{e}$$

$$-4\pi e n_0 \rho(\varphi_\mathrm{i}) = k_\rho^2\,\varphi$$

$$k_\rho(\varphi_\mathrm{i}) = \lambda_\mathrm{Di}^{-1}\sqrt{\rho(\varphi_\mathrm{i})/\varphi_\mathrm{i}}$$

$$k_\rho \approx k_* + (k_0 - k_*)\tanh(c_* y)$$

$$k_0 = \lambda_\mathrm{D}^{-1} \qquad y = r - a$$

Appendix D

V. Tsytovich, N. Gusein-Zade, and G. Morfill, IEEE Trans. Plasma Sci. **32**, 637 (2004)

In the nonlinear regime one can write the following model equation for the potential near the grain

$$\Delta\varphi_{\rm i} = 2\sqrt{\varphi_{\rm i}/\pi} \tag{1}$$

where $\varphi_{\rm i} = -e\varphi/kT_{\rm i}$ and r is normalized to $\lambda_{\rm Di}$

Introducing the screening factor φ_s as

$$\varphi_{\rm i} = \varphi_{\rm i}(a)\,(a/r)\,\varphi_s,$$

we obtain from (1)

$$\partial^2\varphi_s/\partial r^2 = 2\,(\pi\beta)^{-1/2}\sqrt{\varphi_s r} \tag{2}$$

If we consider (2) in the region near $\lambda_{\rm Di}$, i.e., at $r \sim 1$,
we obtain the following solution

$$\varphi_s = [1 - k_s(r-1)]^4\,, \qquad k_s = (36\pi\beta)^{-1/4}, \qquad \lambda_s \sim \beta^{1/4}$$